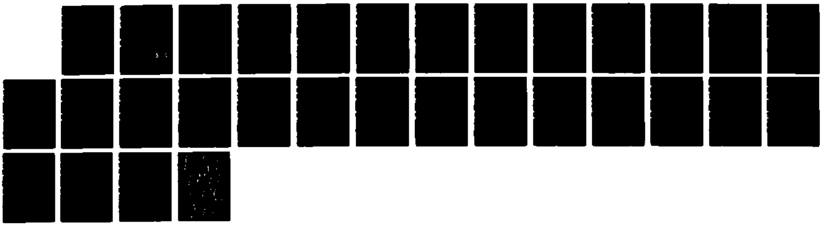
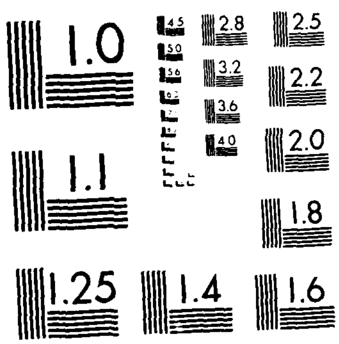
AD-A193 696 DIVERGENCE STABILITY IN CONNECTION WITH THE P-VERSION OF THE FINITE ELEME (U) MARYLAND UNIV COLLEGE PARK DEPT OF MATHEMATICS S JENSEN ET AL NOV 87 1/1

UNCLASSIFIED MD87-47-SJ/MV N00014-87-K-0427 F/G 12/1 NL




1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0

DIVERGENCE STABILITY IN CONNECTION WITH THE P-VERSION OF THE FINITE ELEMENT METHOD

S. Jensen
Department of Mathematics
University of Maryland
Baltimore, MD 21228

M. Vogelius
Department of Mathematics
University of Maryland
College Park, MD 20742

MD87-47-SJ/MV

TR87-47

November 1987

This research was partially supported by ONR contract N00014-87-K-0427 (S.J.), ONR contract N00014-85-K-0169, NSF grant DMS-8601490 and the Sloan Foundation (M.V.).

Encl (1)

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
MD87-47-SJ/MV
TR87-47

2. GOVT ACCESSION NO.
696

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
Divergence Stability in Connection with the P-Version of the Finite Element Method

5. TYPE OF REPORT & PERIOD COVERED
Final life of the contract

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
S. Jensen[1] and M. Vogelius[2]

8. CONTRACT OR GRANT NUMBER(s)
[1]ONR N00014-87-K-0427
[2]ONR N00014-85-K-0169
NSF DMS-8601490 & Sloan Found.

9. PERFORMING ORGANIZATION NAME AND ADDRESS
University of Maryland
Department of Mathematics
College Park, MD 20742

Department of Math
Baltimore, MD 21228

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
Department of the Navy
Office Naval Research
Arlington, VA 22217

12. REPORT DATE
November 1987

13. NUMBER OF PAGES
26

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The paper analyzes the divergence stability of the p-version of the finite element method with the applications to the Stokes problem and elasticity problem with nearly uncompressible material.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Dedicated to J. Douglas, Jr. on the occasion of his 60th birthday.

0. Introduction

Many problems in continuum mechanics involve an incompressibility condition, usually in the form of a divergence constraint. The numerical discretization of such a constraint presents some interesting problems with regard to stability. As an important example we consider the two-dimensional Stokes equations,

$$
(1)\qquad \begin{aligned} -\Delta \underline{U} + \nabla P &= \underline{F} \quad \text{in } \Omega \subseteq \mathbb{R}^2, \\ \nabla\cdot\underline{U} &= 0 \quad \text{in } \Omega, \end{aligned}
$$

with appropriate boundary conditions on $\partial\Omega$. This has the standard weak formulation

$$
(2)\qquad \begin{aligned} &\text{Find } \underline{U} \in \mathcal{V} \subseteq [H^1(\Omega)]^2 \text{ and } P \in \mathcal{W} \subseteq L^2(\Omega) \text{ such that} \\ &a(\underline{U},\underline{v}) + b(\underline{v},P) = (F,\underline{v}) \quad \forall\ \underline{v} \in \mathcal{V} \\ &b(\underline{U},q) = 0 \qquad \forall\ q \in \mathcal{W}. \end{aligned}
$$

The bilinear forms a and b are given by

$$
a(\underline{U},\underline{v}) = 2\int_\Omega \sum_{i,j} \varepsilon_{ij}(\underline{U})\varepsilon_{ij}(\underline{v})\, d\underline{x}
$$

$$
b(\underline{v},P) = -\int_\Omega \nabla\cdot\underline{v}\ P\, d\underline{x},
$$

and $(\underline{F},\underline{v})$ denotes the usual $[L^2(\Omega)]^2$ inner product. The tensor $\varepsilon_{ij}(\underline{v})$ is the symmetric derivative $\frac{1}{2}\left(\frac{\partial}{\partial x_i}v_j + \frac{\partial}{\partial x_j}v_i\right)$. The spaces $\mathcal{V}$ and $\mathcal{W}$ depend on the boundary conditions. For no-slip boundary conditions: $\mathcal{V} = [\mathring{H}^1(\Omega)]^2$ and $\mathcal{W} = L^2(\Omega) \cap \left\{\int_\Omega q = 0\right\}$, for stress-free boundary conditions: $\mathcal{V}$ = the orthogonal

complement of $\{\varepsilon_{ij}(\underline{v}) = 0\}$ in $[H^1(\Omega)]^2$ and $\mathscr{W} = L^2(\Omega)$. A natural discretization of (2) consists in choosing finite dimensional spaces $\mathscr{V}_N \subseteq \mathscr{V}$, $\mathscr{W}_N \subseteq \mathscr{W}$ and determining

$$\underline{U}_N \in \mathscr{V}_N \text{ and } P_N \in \mathscr{W}_N \text{ such that}$$

$$\text{(3)} \qquad \begin{aligned} a(\underline{U}_N,\underline{v}) + b(\underline{v},P_N) &= (\underline{F},\underline{v}) \quad \forall \ \underline{v} \in \mathscr{V}_N \\ b(\underline{U}_N,q) &= 0 \qquad \forall \ q \in \mathscr{W}_N. \end{aligned}$$

The main obstacle in connection with (3) is to find spaces $\mathscr{V}_N$ and $\mathscr{W}_N$ so that the discretization is stable and at the same time has good approximation properties. A reasonable requirement concerning stability seems to be

$$\text{(4)} \qquad \|\underline{U} - \underline{U}_N\|_{H^1} + \|P - P_N\|_{L^2} \le C\left[\min_{\underline{v}\in\mathscr{V}_N} \|\underline{U} - \underline{v}\|_{H^1} + \min_{q\in\mathscr{W}_N} \|P - q\|_{L^2}\right],$$

with C independent of the dimension variable N. It is well known that the Babuska-Brezzi condition

$$\text{(5)} \qquad \min_{q\in\mathscr{W}_N\setminus\{0\}} \ \max_{\underline{v}\in\mathscr{V}_N} \frac{\int_\Omega \nabla\cdot\underline{v} \ q \ d\underline{x}}{\|\underline{v}\|_{H^1}\|q\|_{L^2}} \ge c > 0,$$

with c independent of N, is sufficient to quarantee (4) (cf. [2], [7]). If the pressure spaces $\mathscr{W}_N$ are chosen equal to $\nabla\cdot\mathscr{V}_N$, then (5) is equivalent to the requirement that the divergence operator

$$\nabla\cdot \ : \mathscr{V}_N \longrightarrow \mathscr{W}_N$$

has corresponding right inverses

$$(\nabla\cdot)_N^{-1} \ : \mathscr{W}_N \longrightarrow \mathscr{V}_N,$$

that are uniformly bounded in $\mathscr{B}(L^2;H^1)$. In this case (5) is both

a necessary and sufficient condition that the quasi-optimality estimate

$$\|\underline{U} - \underline{U}_N\|_{H^1} \leq C \min_{\underline{v} \in \mathcal{V}_N} \|\underline{U} - \underline{v}\|_{H^1}$$

holds for arbitrary admissible force F (cf. [13]).

The most natural low degree finite element spaces often fail to satisfy the Babuska-Brezzi condition, except on very special triangulations. A remedy is to appropriately enlarge the velocity space or to deplete the pressure space; such approaches are analyzed for continuous piecewise linear (bilinear) velocities with piecewise constant pressures in [6] and [11] respectively. For continuous piecewise quadratic velocities one has the well known Taylor-Hood element, with continuous piecewise linear pressures (for the analysis leading to (5), see [4] and [14]). Enlarging the velocity space or depleting the pressure space is also in general necessary for cubic velocities and quadratic pressures (cf. [13]).

For continuous piecewise polynomial velocities of total degree $\leq p$, $p \geq 4$, the situation is quite different. For an arbitrary triangulation the range of the divergence operator acting on the velocity space has a very simple characterization -- it consists of all piecewise polynomials of total degree $\leq p - 1$, except for a certain constraint at so-called singular vertices (cf. [12]). Furthermore, for fixed $p \geq 4$, the divergence operator possesses maximal right inverses, the norms of which are bounded independently of the mesh size h. To paraphrase: the condition (5) is satisfied for such velocities if the pressure space, $\mathcal{V}_N$,

is chosen to be $\nabla\cdot\mathcal{V}_N$. Using the analysis in [15] we are able to prove that the same right inverses have $\mathcal{B}(L^2;H^1)$ norms, which are bounded by some power of p, for fixed h. These results $(p \geq 4)$ are valid also for piecewise polynomial spaces with a homogeneous Dirichlet condition on $\partial\Omega$ (assuming, of course, Ω is polygonal).

In this note we show with a few examples, theoretical as well as computational, that it is <u>not</u> in general possible to find maximal right inverses for the divergence operator, acting on entire polynomials of degree $\leq p$, the norms of which are bounded in $\mathcal{B}(L^2;H^1)$, uniformly in p. We discuss both spaces of total and separate degree $\leq p$, as well as spaces with and without boundary conditions.

The lack of uniformly bounded right inverses for the discrete case is somewhat surprising when compared to the continuous case: it is easy to see that there exists a right inverse $(\nabla\cdot)^{-1}$ which maps $H^s = \nabla\cdot(H^{s+1})^2$ boundedly into $(H^{s+1})^2$, $\forall\ s \geq 0$. A similar result holds with homogeneous Dirichlet boundary conditions, even for non-smooth (polygonal) domains Ω (cf. [1]).

Methods that use high degree polynomials to approximate the solution to the Stokes equations are quite common, whether they be variationally based spectral methods, or collocation based pseudo-spectral methods (cf. [10]). Another possibility is the so-called p-version of the finite element method (cf. [3]): it uses a rather coarse mesh (triangulation or lattice) and achieves convergence by including, in a variational formulation, piecewise polynomials of high degree relative to this mesh. Even though the Babuska-Brezzi

condition may only be satisfied with a constant approaching zero as some negative power of p, these methods seem to have (nearly) optimal convergence rates as far as the velocity is concerned. We briefly return to an explanation of this (at least for variational methods) towards the very end of this paper. In the special case of periodic boundary conditions it is normal to consider spectral or pseudospectral methods based on trigonometric polynomials instead of polynomials. The resulting methods are much more likely to be uniformly divergence stable (see, e.g., [8]), however, they are restricted in their applicability due to the boundary conditions.

To complete the introduction, let us give an interpretation of the constant

$$\min_{q \in \mathcal{V}_N \setminus \{0\}} \max_{\underline{v} \in \mathcal{Y}_N} \frac{\int_\Omega \nabla \cdot \underline{v}\ q\ d\underline{x}}{\|\underline{v}\|_{H^1} \|q\|_{L^2}} = \mu_N,$$

in terms of the associated matrices, when $\mathcal{V}_N = \nabla \cdot \mathcal{Y}_N$. In order to do so, we first need to specify our choice of norm on $[H^1]^2$: of the many equivalent norms we take

$$\|\underline{v}\|_{H^1} = \left[\sum_{i=1}^{2} \int_\Omega \left| \frac{\partial}{\partial x_i} \underline{v} \right|^2 d\underline{x} + \left| \int_\Omega \underline{v}\ d\underline{x} \right|^2 \right]^{1/2}.$$

Let $\{\underline{\varphi}_\ell\}_{\ell=1}^N$ be a basis for $\mathcal{Y}_N$ and let $\{\psi_k\}_{k=1}^M$ be a basis for $\mathcal{V}_N = \nabla \cdot \mathcal{Y}_N$. The matrices $A = (a_{k\ell})_{k=1,\ell=1}^{M,N}$, $B = (b_{k\ell})_{k=1,\ell=1}^{N,N}$ and $C = (c_{k\ell})_{k=1,\ell=1}^{M,M}$ are defined by

$$\text{(6.a)} \qquad \nabla \cdot \underline{\varphi}_\ell = \sum_{k=1}^{M} a_{k\ell}\ \psi_k, \quad 1 < \ell \le N,$$

(6.b) $$b_{k\ell} = \sum_{i=1}^{2} \int_\Omega \frac{\partial}{\partial x_i} \underline{\varphi}_k \cdot \frac{\partial}{\partial x_i} \underline{\varphi}_\ell \, d\underline{x} + \int_\Omega \underline{\varphi}_k \, d\underline{x} \cdot \int_\Omega \underline{\varphi}_\ell \, d\underline{x},$$

$1 \le k,\ell \le N,$ and

(6.c) $$c_{k\ell} = \int_\Omega \psi_k \psi_\ell \, d\underline{x}, \quad 1 \le k,\ell \le M.$$

A is the discrete representation of the divergence operator and $(B\cdot,\cdot)$ and $(C\cdot,\cdot)$ represent the quadratic forms $\|\underline{v}\|^2_{H^1}$ and $\|q\|^2_{L^2}$ respectively.

With these definitions it is easy to see that μ_N is the smallest singular value of the $N \times M$ matrix $B^{-1/2}A^T C^{1/2}$, and this in turn is the square root of the smallest eigenvalue of the positive definite symmetric $M \times M$ matrix

(7) $$D = C^{1/2} A B^{-1} A^T C^{1/2}.$$

For any $q \in \Psi_N$ let $(\nabla\cdot)_N^{-1} q \in \mathcal{V}_N$ denote the element of minimal H^1-norm that has q for its divergence. By a "worst possible pressure" (as far as divergence stability is concerned) we mean a $q_0 \in \Psi_N$, $\|q_0\|_{L^2} = 1$, on which the "minimal norm" right inverse $(\nabla\cdot)_N^{-1}$ attains its operator norm. If $\underline{x} \in \mathbb{R}^M$ denotes a unit eigenvector for the matrix D, corresponding to the smallest eigenvalue, μ_N, then

(8) $$q_0 = \sum_{j=1}^{M} (C^{-1/2}\underline{x})_j \, \psi_j \in \Psi_N$$

is a worst possible pressure.

1. Theoretical and Computational results.

Let R denote the square $(-1,1) \times (-1,1)$. We first consider polynomials of separate degree $\le p$, i.e., the velocity space is $(Q_p)^2$ where

(9) $$Q_p = \mathrm{span}\{x_1^m x_2^n : 0 \le m,n \le p\},$$

and the corresponding pressure space is

(10) $$\nabla \cdot (Q_p)^2 = \mathrm{span}\{x_1^m x_2^n : 0 \le m,n \le p, \quad m + n < 2p\}.$$

Note that we use p as a subscript instead of the dimension variable $N = (p + 1)^2$. As before $(\nabla \cdot)_p^{-1}$ denotes the right inverse with minimal $H^1(R)$ norm.

Proposition 1.

The operator $(\nabla \cdot)_p^{-1} : \nabla \cdot (Q_p)^2 \longrightarrow (Q_p)^2$, $p \ge 1$, considered as an operator from a subspace of $L^2(R)$ to a subspace of $(H^1(R))^2$ satisfies

$$cp \le \|(\nabla \cdot)_p^{-1}\|_{\mathcal{B}(L^2;H^1)} \le Cp,$$

with constants $0 < c$ and C independent of p.

Before giving a proof of Proposition 1 we make a few observations about orthogonal polynomials. Let $\ell_n(x)$ denote the Legendre polynomial of degree n, with the standard normalization

(11) $$\int_{-1}^{1} \ell_n^2 \, dx = \frac{2}{2n+1}.$$

It is not difficult to see that

$$(12) \qquad \int_{-1}^{x} \ell_n = \frac{1}{2n+1}(\ell_{n+1}(x) - \ell_{n-1}(x)), \quad n \geq 1.$$

The polynomial ℓ_n may be written as a telescoping series

$$\ell_n = \sum_{j=0}^{[n/2]-1} (\ell_{n-2j} - \ell_{n-2(j+1)}) + \begin{cases} \ell_1, & n \text{ odd} \\ \ell_0, & n \text{ even} \end{cases},$$

and consequently

$$\ell_n(x) = \sum_{j=0}^{[n/2]-1} (2(n-2j-1)+1) \int_{-1}^{x} \ell_{n-2j-1} + \begin{cases} \ell_1, & n \text{ odd} \\ \ell_0, & n \text{ even} \end{cases}.$$

From this we conclude that $\int_{-1}^{1} \left(\frac{d}{dx}\ell_n\right)^2 dx$ is of the order

$\sum_{j=0}^{[n/2]-1} j$, i.e.,

$$(13) \qquad \int_{-1}^{1} \left(\frac{d}{dx}\ell_n\right)^2 dx \text{ is of the order } n^2.$$

Let $q_n(x)$, $0 \leq n \leq p$ denote the polynomials

$$(14) \qquad \begin{aligned} q_n(x) &= \ell_n(x), \quad 0 \leq n < p \\ q_p(x) &= \int_{-1}^{x} \ell_{p-1} \end{aligned}$$

(the notation should properly be $q_n^{(p)}$, since the definition of q_n depends on p, but we drop the superscript for convenience). A simple computation gives that the polynomial

$$r(x) = \alpha\ell_{p-2} + \beta \int_{-1}^{x} \ell_{p-1}, \quad p \geq 2,$$

satisfies

$$\frac{2}{3}\left[\frac{\alpha^2}{2p+1} + \frac{\beta^2}{(2p+1)(2p-1)^2}\right] \le \int_{-1}^{1} (r(x))^2\, dx \le 6\left[\frac{\alpha^2}{2p-3} + \frac{\beta^2}{(2p-3)(2p-1)^2}\right].$$

Since $\{\ell_n\}_{n=1}^{p-3} \cup \{\ell_{p-1}\}$ are mutually orthogonal in L^2, and since they are also orthogonal to ℓ_{p-2} and $\int_{-1}^{x} \ell_{p-1}$, it follows that

$$\int_{-1}^{1} \left[\sum_{n=0}^{p} \alpha_n q_n(x)\right]^2 dx \quad \text{is equivalent to}$$

(15)

$$\sum_{n=0}^{p-1} \alpha_n^2 (n+1)^{-1} + \alpha_p^2 (p+1)^{-3}, \quad \text{with constants}$$

that are independent of p.

It is convenient to work with

$$\{q_m(x_1) q_n(x_2) \;:\; 0 \le m,n \le p, \quad m + n < 2p\}$$

as a basis for $\nabla\cdot(\mathcal{Q}_p)^2$. Based on (15) we get that if

$$q = \sum_{\substack{m,n=0 \\ m+n<2p}}^{p} \alpha_{mn} q_m(x_1) q_n(x_2),$$

then

$$\int_{\Omega} (q)^2 d\underline{x} \quad \text{is equivalent to}$$

(16)

$$\sum_{\substack{m<p \\ n<p}} (\alpha_{mn})^2 (m+1)^{-1} (n+1)^{-1} + \sum_{m<p} ((\alpha_{mp})^2 + (\alpha_{pm})^2)(m+1)^{-1} p^{-3},$$

with constants that are independent of p.

We are now ready for the proof of proposition 1:

Given $q \in \nabla\cdot(\mathcal{Q}_p)^2$ we have

$$q = \sum_{\substack{m,n=0 \\ m+n<2p}}^{p} \alpha_{mn} q_m(x_1) q_n(x_2)$$

for some set of coefficients $\{\alpha_{mn}\}$. Define

$$\underline{u} = \begin{bmatrix} \displaystyle\sum_{n<m<p} \alpha_{mn}\left[\int_{-1}^{x_1} q_m\right] q_n(x_2) + \sum_{m<p} \alpha_{mp}\left[\int_{-1}^{x_1} q_m\right] q_p(x_2) \\ \\ \displaystyle\sum_{m\le n<p} \alpha_{mn}\, q_m(x_1)\int_{-1}^{x_2} q_n + \sum_{n<p} \alpha_{pn} q_p(x_1) \int_{-1}^{x_2} q_n \end{bmatrix}$$

It is clear that $\underline{u} \in (\mathcal{Q}_p)^2$, with $\nabla\cdot\underline{u} = q$. It remains to estimate the H^1-norm of $\underline{u}$. Using (16) we immediately get that

$$\left\|\frac{\partial}{\partial x_1}u_1\right\|_{L^2(R)} + \left\|\frac{\partial}{\partial x_2}u_2\right\|_{L^2(R)} \le C\|q\|_{L^2(R)};$$

also

$$\int_R \underline{u} = -\frac{4}{3}\begin{bmatrix} \alpha_{10} \\ \alpha_{01} - 3\alpha_{00} \end{bmatrix}, \quad \text{for} \quad p > 2,$$

and therefore

$$\left| \int_R \underline{u} \right| \le C\|q\|_{L^2(R)}.$$

Concerning $\frac{\partial}{\partial x_2}u_1$:

$$\left\|\frac{\partial}{\partial x_2}u_1\right\|^2_{L^2(R)}$$

$$(17) \qquad = \left\| \sum_{n<m<p} \alpha_{mn}\left(\int_{-1}^{x_1} \ell_m\right)\left(\frac{d}{dx}\ell_n\right)(x_2) + \sum_{m<p} \alpha_{mp}\left[\int_{-1}^{x_1} \ell_m\right]\ell_{p-1}(x_2)\right\|^2_{L^2(R)}$$

$$\le C\sum_{m<p}(m+1)^{-3}\left\|\sum_{n=0}^{m-1}\alpha_{mn}\left(\frac{d}{dx}\ell_n\right) + \alpha_{mp}\ell_{p-1}\right\|^2_{L^2(-1,1)},$$

because of the identities (11) and (12) and the L^2-orthogonality of the Legendre polynomials. From (17) we get by means of the triangle inequality, Schwarz inequality, and the estimate (13)

$$\left\|\frac{\partial}{\partial x_2} u_1\right\|^2_{L^2(R)} \le C \sum_{m<p} (m+1)^{-3} \left[\sum_{n=0}^{m-1} (\alpha_{mn})^2 \cdot \sum_{n=0}^{m-1} n^2 + (\alpha_{mp})^2 p^{-1} \right],$$

the right hand side of which is bounded by

$$C \left\{ \sum_{n<m<p} (\alpha_{mn})^2 + \sum_{m<p} (\alpha_{mp})^2 (m+1)^{-3} p^{-1} \right\}.$$

Using the above estimate in combination with (16) we get

$$\left\|\frac{\partial}{\partial x_2} u_1\right\|^2_{L^2(R)} \le Cp^2 \, \|q\|^2_{L^2(R)} .$$

The same estimate holds for $\left\|\frac{\partial}{\partial x_1} u_2\right\|^2_{L^2(R)}$. In summary we have thus established

$$\|\underline{u}\|_{H^1(R)} \le Cp \, \|q\|_{L^2(R)} ,$$

and since $(\nabla\cdot)_p^{-1} q$ is the field of minimal H^1-norm it follows that

$$\|(\nabla\cdot)_p^{-1}\|_{\mathcal{B}(L^2;H^1)} \le Cp.$$

To verify the second inequality of this proposition, consider

$$q^* = r(x_1)\ell_p(x_2)$$

for some fixed polynomial r (of degree $\le p-1$). As a basis for $\mathcal{Q}_p$ we choose

$$\{q_m(x_1)q_n(x_2) \; : \; 0 \le m,n \le p\}.$$

For an arbitrary $\underline{u} \in (Q_p)^2$ there exist coefficients (α_{mn}) and (β_{mn}) such that

$$u_1 = \sum_{m,n=0}^{p} \alpha_{mn} q_m(x_1) q_n(x_2)$$

$$u_2 = \sum_{m,n=0}^{p} \beta_{mn} q_m(x_1) q_n(x_2).$$

If $\nabla \cdot \underline{u} = \frac{\partial}{\partial x_1} u_1 + \frac{\partial}{\partial x_2} u_2 = q^*$, then we must necessarily have

$$\sum_{m=0}^{p} \alpha_{mp} \left[\frac{d}{dx} q_m\right](x_1) \int_{-1}^{1} q_p \ell_p \, dx$$

$$= \int_{-1}^{1} \nabla \cdot \underline{u}(x_1, x_2) \ell_p(x_2) dx_2 = r(x_1) \int_{-1}^{1} \ell_p^2 dx.$$

Due to (12) and (14) it follows that

$$\frac{d}{dx}\left[\sum_{m=0}^{p} \alpha_{mp} q_m\right] = (2p - 1) r,$$

and therefore

$$(18) \qquad u_1 = \sum_{n<p} q_n(x_2) \sum_{m \le p} \alpha_{mn} q_m(x_1) + q_p(x_2)(2p-1)\left(\int_{-1}^{x_1} r + c\right).$$

After differentiation with respect to x_2

$$\frac{\partial}{\partial x_2} u_1 = \sum_{n<p} \left[\frac{d}{dx} q_n\right](x_2) \sum_{m \le p} \alpha_{mn} q_m(x_1) + \ell_{p-1}(x_2)(2p-1)\left(\int_{-1}^{x_1} r + c\right),$$

from which it now follows (by orthogonality) that

$$(19) \qquad \int_R \left[\frac{\partial}{\partial x_2} u_1\right]^2 d\underline{x} \ \geq \ 2(2p-1) \int_{-1}^{1} \left(\int_{-1}^{x} r + c\right)^2 dx$$

$$\geq \ 2(2p-1) \left\| \int_{-1}^{x} r - \frac{1}{2} \int_{-1}^{1}\int_{-1}^{x} r \ dx \right\|^2_{L^2(-1,1)} .$$

At the same time

$$\|q^*\|^2_{L^2(R)} = \frac{2}{2p+1} \|r\|^2_{L^2(-1,1)} ,$$

and so we have proven that for any $\underline{u} \in \mathcal{Q}_p$ with $\nabla \cdot \underline{u} = q^*$ one has

$$\|\underline{u}\|_{H^1(R)} \geq cp \|q^*\|_{L^2(R)}$$

(for fixed r). This verifies the lower bound on the norm of $(\nabla\cdot)_p^{-1}$.

■

We now consider polynomials of total degree $\leq p$. Without boundary conditions the velocity space is $(\mathcal{P}_p)^2$, where

$$\mathcal{P}_p = \mathrm{span}\{x_1^m x_2^n : m + n \leq p\},$$

and the corresponding pressure space is $\nabla\cdot(\mathcal{P}_p)^2 = \mathcal{P}_{p-1}$. First some computational results for the domain $R = (-1,1) \times (-1,1)$.

As a basis for $\mathcal{P}_p$ in these computations we pick products of integrals of Legendre polynomials (supplemented by the constant function):

$$(20) \qquad \begin{array}{l} 1, \quad \int_{-1}^{x_1} \ell_m, \quad \int_{-1}^{x_2} \ell_n \qquad 0 \leq m,n \leq p-1, \\[2ex] \int_{-1}^{x_1} \ell_m \int_{-1}^{x_2} \ell_n \qquad 0 \leq m,n \ \text{ and } \ m+n \leq p-2. \end{array}$$

As a basis for $\nabla\cdot(\mathcal{P}_p)^2 = \mathcal{P}_{p-1}$ we pick

(21) $\qquad \ell_m(x_1)\ell_n(x_2) \qquad 0 \le m,n$ and $m+n \le p-1.$

The top plot in figure 1 shows the smallest eigenvalue of the matrix D (as defined in (7)) for p varying between 1 and 18. The eigenvalues were computed using two EISPACK subroutines: first the matrix was transformed (by orthogonal similarity transformations) into a tridiagonal matrix using subroutine TRED2, then the eigenvalues were computed by the QL method (an obvious variant of the QR method) using subroutine TQL2. The $\mathcal{B}(L^2;H^1)$ norm of the "minimal norm" right inverse is the reciprocal square root of the smallest eigenvalue. The numbers do not clearly indicate whether these right inverses are bounded independently of p -- if anything they seem to indicate that the norms grow as $p \longrightarrow \infty$, but only as a very small power or possibly a logarithm of p (the corresponding solid line was computed by linear regression on the last four points, it is proportional to $p^{-0.3975}$). We note that by a slight change in the proof of Proposition 1 one can show that the present right inverses are bounded in $\mathcal{B}(L^2;H^1)$ by Cp, however, that is clearly too conservative. The situation is more clear cut in the case when homogeneous Dirichlet boundary conditions are prescribed. The velocity space is then $(\mathring{\mathcal{P}}_p)^2$, where

$$\mathring{\mathcal{P}}_p = \mathcal{P}_p \cap \{v|_{\partial\Omega} = 0\}$$

and the pressure space is $\nabla\cdot(\mathring{\mathcal{P}}_p)^2 \subseteq \mathcal{P}_{p-1}$. A simple count of dimensions gives that $\nabla\cdot(\mathring{\mathcal{P}}_p)^2$ has co-dimension 9 in $\mathcal{P}_{p-1}$, $p \ge 5$. $\nabla\cdot(\mathring{\mathcal{P}}_p)^2$ is the common nullspace for the following nine

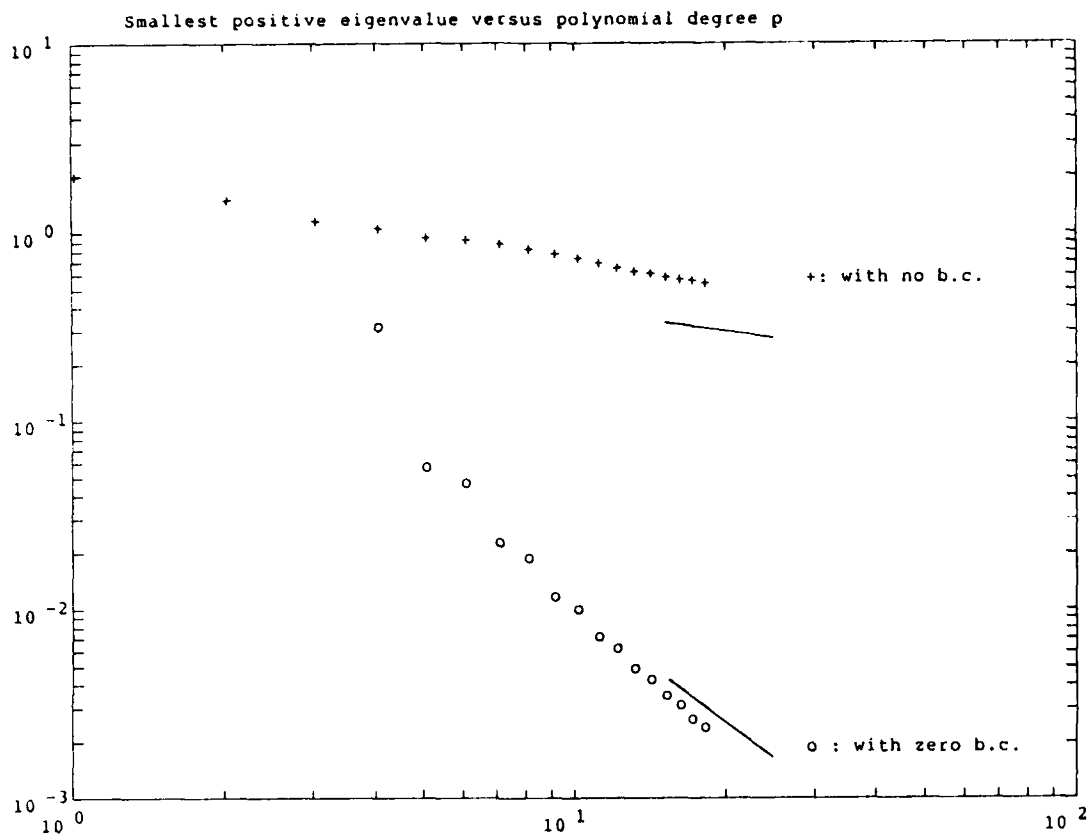

Smallest positive eigenvalue versus polynomial degree p
+: with no b.c.
o : with zero b.c.
10 1
10 0
10 -1
10 -2
10 -3
10 0
10 1
10 2

Figure 1

linearly independent functionals: the integral over R, point evaluation at each of the corners of ∂R and $\left(\frac{\partial}{\partial n}\right)^{p-2}$ evaluated at the center of each side of ∂R. As a basis for $\mathring{\mathcal{P}}_p$ we take the same elements as in (20) except for those in the first line, and those corresponding to $m = 0$ or $n = 0$ in the second line. Instead of computing the matrices A and C by using a basis for $\nabla\cdot(\mathring{\mathcal{P}}_p)^2$ we use a basis for $\mathcal{P}_{p-1} \cap \left\{ \int_\Omega q = 0\right\}$, the same as in (21) except for the constant function. The only effect of this in terms of eigenvalues and eigenvectors is to add 0 as an eigenvalue of D with multiplicity 8, $p \geq 5$ (in the case $p = 4$, $\nabla\cdot(\mathring{\mathcal{P}}_p)^2$ only has codimension 8 in $\mathcal{P}_{p-1}$, and the corresponding multiplicity of 0 becomes 7). The lower plot in figure 1 shows the smallest positive eigenvalue of D for p varying between 4 and 18; the eigenvalues were computed as before, using EISPACK. The $\mathcal{B}(L^2;H^1)$ norm of the "minimal norm" right inverse $(\nabla\cdot)_p^{-1}$: $\nabla\cdot(\mathring{\mathcal{P}}_p)^2 \longrightarrow (\mathring{\mathcal{P}}_p)^2$ is the reciprocal square root of the smallest positive eigenvalue. The numbers clearly indicate that the norms are not bounded uniformly in p. Based on these numbers it is reasonable to conjecture that the norms grow at least linearly in p and at most as $Cp^{3/2}$ (the solid line on the graph of eigenvalues is proportional to p^{-2}, corresponding to linear growth of the norms). What is more, we can actually prove the lower bound:

Proposition 2.

Let $(\nabla\cdot)_p^{-1}: \nabla\cdot(\mathring{\mathcal{P}}_p)^2 \longrightarrow (\mathring{\mathcal{P}}_p)^2$, $p \ge 4$, denote the right inverse with minimal H^1 norm. Considered as an operator from a subspace of $L^2(R)$ to a subspace of $(H^1(R))^2$ this satisfies

$$cp \le \|(\nabla\cdot)_p^{-1}\|_{\mathcal{B}(L^2;H^1)},$$

with $c > 0$, independent of p.

Proof.

Let $q^*(x_1,x_2) = \ell_1(x_1)\int_{-1}^{x_2}\ell_{p-3}$. It is clear that $q^* \in \nabla\cdot(\mathring{\mathcal{P}}_p)^2$ since $\nabla\cdot\underline{v} = q^*$ for $\underline{v} = (\int_{-1}^{x_1}\ell_1 \int_{-1}^{x_2}\ell_{p-3}, 0)$. On the other hand, if $\underline{u} \in (\mathring{\mathcal{P}}_p)^2$ is an arbitrary velocity field with $\nabla\cdot\underline{u} = q^*$, then

$$\underline{u} = \begin{bmatrix} \displaystyle\sum_{m,n=1}^{m+n\le p-2} \alpha_{mn}\int_{-1}^{x_1}\ell_m \int_{-1}^{x_2}\ell_n \\ \\ \displaystyle\sum_{m,n=1}^{m+n\le p-2} \beta_{mn}\int_{-1}^{x_1}\ell_m \int_{-1}^{x_2}\ell_n \end{bmatrix}$$

and

$$\sum_{m,n=1}^{m+n\le p-2}\left[\alpha_{mn}\ell_m(x_1)\int_{-1}^{x_2}\ell_n + \beta_{mn}(\int_{-1}^{x_1}\ell_m)\ell_n(x_2)\right] = \ell_1(x_1)\int_{-1}^{x_2}\ell_{p-3}.$$

Using (12) and the linear independence of the Legendre polynomials we get

$$(22) \qquad \sum_{n=1}^{p-3} \alpha_{1n} \int_{-1}^{x_2} \ell_n - \sum_{n=1}^{p-4} \frac{1}{5} \beta_{2n} \ell_n(x_2) = \int_{-1}^{x_2} \ell_{p-3},$$

since these are the respective coefficients of $\ell_1(x_1)$. The identity (22) implies

$$\alpha_{1p-3} = 1,$$

and since

$$\frac{\partial}{\partial x_2} u_1 = \sum_{m,n=1}^{m+n\le p-2} \alpha_{mn} \left(\int_{-1}^{x_1} \ell_m\right) \ell_n(x_2)$$

$$= \sum_{m,n=1}^{m+n\le p-2} \frac{1}{2m+1} \alpha_{mn} \Big[\ell_{m+1}(x_1) - \ell_{m-1}(x_1)\Big] \ell_n(x_2),$$

it follows that

$$(23) \qquad \left\|\frac{\partial}{\partial x_2} u_1\right\|^2_{L^2(R)} \ge \frac{1}{9} \alpha^2_{1p-3} \int_{-1}^{1} \ell_0^2 \int_{-1}^{1} \ell^2_{p-3}$$

$$= \frac{4}{9(2p-5)} .$$

A simple computation gives

$$\|q^*\|^2_{L^2(R)} = \frac{8}{3} \frac{1}{(2p-3)(2p-5)(2p-7)} ,$$

and therefore, in light of (23),

$$\|\underline{u}\|^2_{H^1(R)} \ge \frac{1}{6} (2p-3)(2p-7) \|q^*\|^2_{L^2(R)} .$$

Since $\underline{u}$ is an arbitrary field in $(\mathring{\mathcal{P}}_p)^2$, with $\nabla \cdot \underline{u} = q^*$, this gives the desired lower bound on the operator norm.

■

Figures 2 and 3 show contour plots of worst possible pressures, in the sense defined at the end of the introduction, for $p = 7, 8, 14$ and 15. These pressures are elements of $\nabla\cdot(\mathring{\mathcal{P}}_p)^2$, and they have the property that the right inverses $(\nabla\cdot)_p^{-1}: \nabla\cdot(\mathring{\mathcal{P}}_p)^2 \longrightarrow (\mathring{\mathcal{P}}_p)^2$ attain their $\mathcal{B}(L^2;H^1)$ norms there. Solid lines in the plots correspond to positive values, dashed lines correspond to negative values. On each plot the interval between contour lines is indicated at the bottom and so is the entire value range of the pressure.

We note several features:

(1) There is a marked difference between worst pressures for even and odd p. This correlates well with the lowest plot in figure 1, which really seems to consist of two slightly different curves, one for even p and one for odd p.

(2) The value range for a worst pressure grows as p increases, and the extreme values are clearly attained on $\partial\Omega$.

(3) For p odd and sufficiently large there is a local checkerboard pattern developing, similar to that found in connection with some unstable low order elements (see, e.g., [5] for bilinear-constant velocity-pressure approximation).

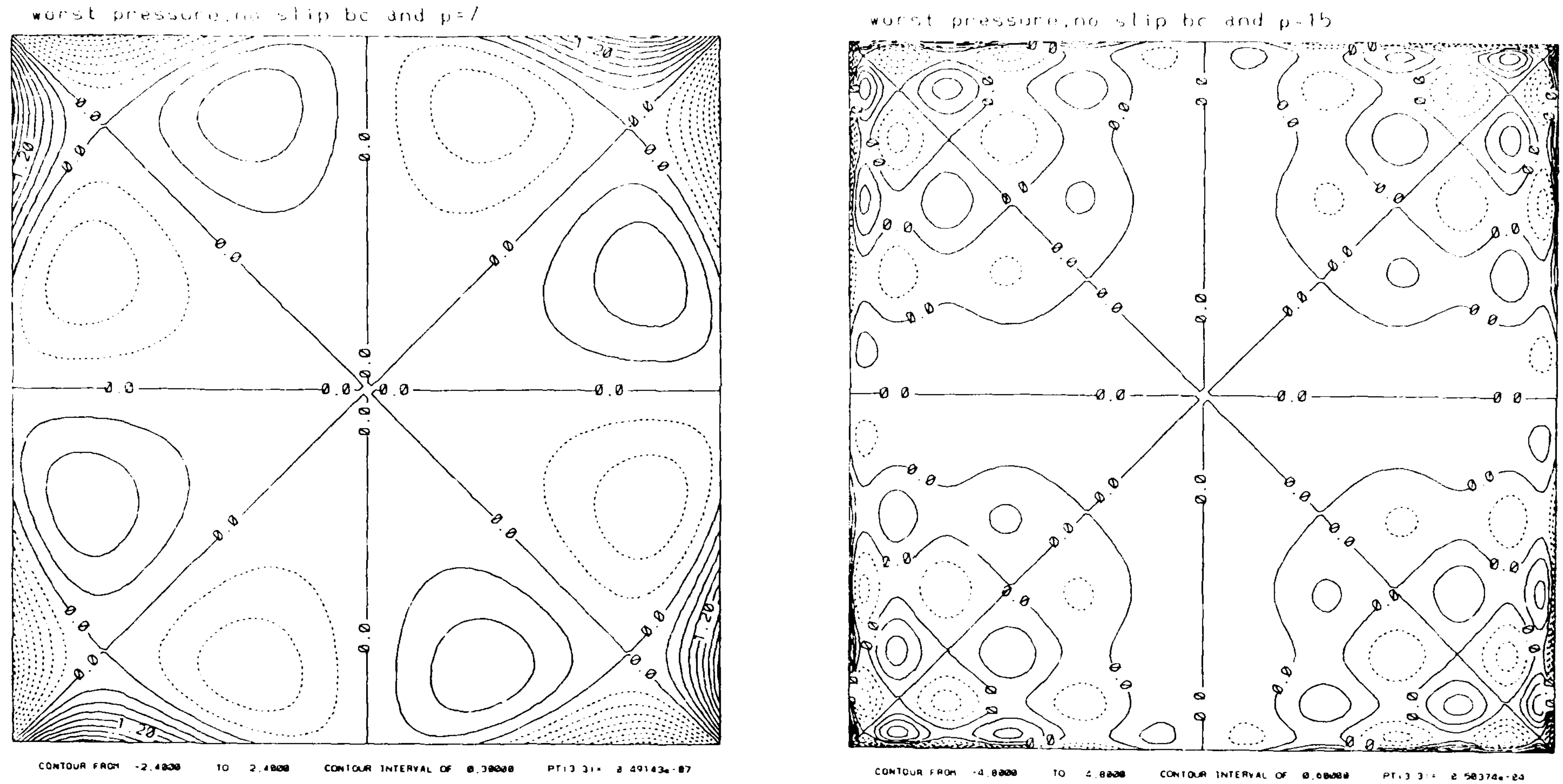

worst pressure,no slip bc and p=7
CONTOUR FROM -2.4000 TO 2.4000 CONTOUR INTERVAL OF 0.30000
worst pressure,no slip bc and p=15
CONTOUR FROM -4.8000 TO 4.8000 CONTOUR INTERVAL OF 0.60000

Figure 2

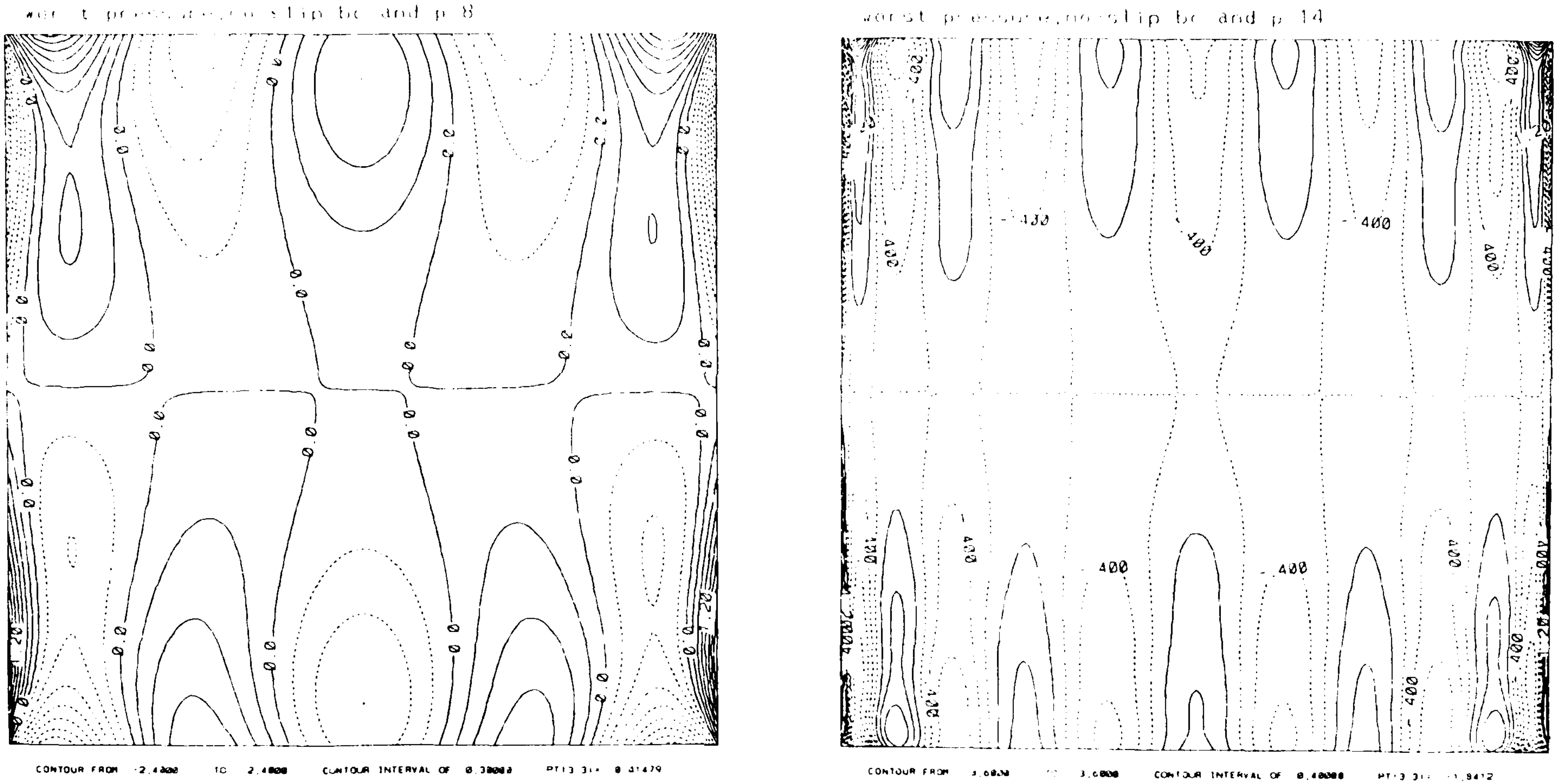

slip bc and p 8
worst pr slip bc and p 14
CONTOUR FROM -2.4000 TO 2.4000 CONTOUR INTERVAL OF 0.30000
CONTOUR INTERVAL OF 0.40000

Figure 3

2. Conclusions and final remarks.

We have shown with a few examples that one cannot in general construct maximal right inverses for the divergence operator, whose $\mathcal{B}(L^2;H^1)$ norms are uniformly bounded as the polynomial degree increases. We do not think that the square domain is extremely special in this regard, and we believe that similar examples can be found on many other polygonal domains.

We do not, however, claim that it is impossible to find domains for which uniformly bounded maximal right inverses exist. Indeed one such class of domains (for polynomials of total degree $\le p$, with no boundary conditions) are the ellipses:

Example 2.1.

Let $E = \{(x_1,x_2): ax_1^2 + bx_2^2 < 1\}$, $0 < a,b$. The Laplace operator $\Delta: \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2$ maps $\mathring{\mathcal{P}}_{p+1}$ into $\mathcal{P}_{p-1}$. The space $\mathring{\mathcal{P}}_{p+1}$, on the domain E, is the same as $(ax_1^2 + bx_2^2 - 1)\mathcal{P}_{p-1}$, and since Δ has no nontrivial null vectors with homogeneous Dirichlet boundary conditions, it follows that Δ is an isomorphism from $\mathring{\mathcal{P}}_{p+1}$ onto $\mathcal{P}_{p-1}$. The operator $(\nabla\cdot)_p^{-1} = \nabla\Delta^{-1}: \mathcal{P}_{p-1} \longrightarrow (\mathcal{P}_p)^2$ is a uniformly bounded maximal right inverse for the divergence operator on the domain E.

■

Where no boundary conditions are involved, unboundedness of the "minimal norm" right inverse in spaces of entire polynomials (on a square, say) immediately leads to unboundedness in spaces of piecewise polynomials relative to a fixed partition (a lattice). We expect that the "minimal norm" right inverses for truly

piecewise polynomials will inherit some of the (likely) extra unboundedness associated with homogeneous boundary conditions. For example, on a lattice (with more than one rectangle) it would not be surprising, if the right inverses corresponding to spaces of piecewise polynomials of separate degree $\le p$ have $\mathcal{B}(L^2;H^1)$ norms, that grow faster than p.

Lack of divergence stability as evidenced by the fact that the best lower bound in (5) might behave like $p^{-\alpha}$, $\alpha > 0$, surprisingly does not lead to suboptimal order of convergence for the velocities, as $p \longrightarrow \infty$ (provided we use the divergence of the velocity space as the pressure space). This may be explained by an interpolation argument, the idea of which originates in [3]. The explanation is particularly simple with stress-free boundary conditions on a simply connected domain: since $\underline{U}_p$ is a projection of $\underline{U}$ we get

$$\|\underline{U} - \underline{U}_p\|_{H^1(\Omega)} \le C\|\underline{U}\|_{H^1(\Omega)}, \tag{24}$$

and since the lower bound in (5) is $\sim p^{-\alpha}$ we can also prove (cf. [13])

$$\|\underline{U} - \underline{U}_p\|_{H^1(\Omega)} \le Cp^{\alpha} \min_{\underline{v}} \|\underline{U} - \underline{v}\|_{H^1(\Omega)},$$

where the minimum is taken over the full space of discrete velocities $((\mathcal{Q}_p)^2$ or $(\mathcal{P}_p)^2)$. Standard approximation results yield

$$\|\underline{U} - \underline{U}_p\|_{H^1(\Omega)} \le C_M p^{-M+\alpha}\|\underline{U}\|_{H^{M+1}(\Omega)}, \quad M \ge 0. \tag{25}$$

For fixed M we can interpolate a fraction $\theta = \frac{k}{M}$ between (24) and (25) to get

$$\|\underline{U} - \underline{U}_p\|_{H^1(\Omega)} \le C_M p^{-k+\alpha k/M} \|\underline{U}\|_{H^{k+1}(\Omega)}, \quad 0 < k < M.$$

By choosing M sufficiently large $(M > \frac{\alpha k}{\varepsilon})$ we conclude that

$$\|\underline{U} - \underline{U}_p\|_{H^1(\Omega)} \le C_{k,\varepsilon} p^{-k+\varepsilon} \|\underline{U}\|_{H^{k+1}(\Omega)} \tag{26}$$

for any $\varepsilon > 0$. Modulo ε the estimate (26) represents the optimal order of convergence for general $\underline{U}$ in $H^{k+1}(\Omega)$. More details are found in [15] and [16], where the same argument is used in the context of the equations of elasticity (for a nearly incompressible material).

We do believe that the lack of divergence stability affects the accuracy of the pressure approximation much more drastically, and we expect that a certain postprocessing (filtering) of the pressures may be necessary as $p \to \infty$.

References

[1] D. N. Arnold, L. R. Scott and M. Vogelius, Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygon. IMA preprint #302, 1987. (to appear, Annali Scuola Norm. Sup. Pisa)

[2] I. Babuska, Error-bounds for the finite element method. Numerische Mathematik, 16 (1971), pp. 322-333.

[3] I. Babuska, B. A. Szabo and I. N. Katz, The p-version of the finite element method. SIAM J. Numer. Anal., 18 (1981), pp. 515-545.

[4] M. Bercovier and O. Pironneau, Error estimates for the finite element method solution of the Stokes problem in the primitive variables. Numerische Mathematik, 33 (1979), pp. 211-224.

[5] J. Boland and R. A. Nicolaides, On the stability of Bilinear-Constant Velocity-Pressure Finite Elements. Numerische Mathematik, 44 (1984), pp. 219-222.

[6] J. Boland and R. A. Nicolaides, Stable and semistable low order finite elements for viscous flows. SIAM J. Numer. Anal., 22 (1985), pp. 474-492.

[7] F. Brezzi, On the existence, uniqueness and approximation of saddlepoint problems arising from Lagrangian multipliers, RAIRO 8 (1974), pp. 129-151.

[8] C. Canuto, Y. Maday and A. Quarteroni, Combined Finite Element and Spectral approximation of the Navier-Stokes equations. Numerische Mathematik, 44 (1984), pp. 201-217.

[9] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. Springer-Verlag, 1986.

[10] D. Gottlieb and S. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications. CBMS-NSF Regional Conference Series in Applied Mathematics, 26. SIAM, 1977.

[11] C. Johnson and J. Pitkäranta, Analysis of some mixed finite element methods related to reduced integration. Math. of Comp., 38 (1982), pp. 375-400.

[12] L. R. Scott and M. Vogelius, Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. Math. Modelling Num. Anal., 19 (1985), pp. 111-143.

[13] L. R. Scott and M. Vogelius, Conforming Finite Element Methods for incompressible and nearly incompressible continua. Lectures in Applied Mathematics, 22, pp. 221-244. AMS, 1985.

[14] R. Verfürht, Error estimates for a mixed finite element approximation of the Stokes equations. RAIRO, Numer. Anal., 18 (1984), pp. 175-182.

[15] M. Vogelius, An analysis of the p-Version of the Finite Element Method for nearly incompressible materials. Uniformly valid, optimal error estimates. Numerische Mathematik, 41 (1983), pp. 39-53.

[16] M. Vogelius, A right inverse for the divergence operator in spaces of piecewise polynomials. Application to the p-Version of the Finite Element Method. Numerische Mathematik, 41 (1983), pp. 19-37.

END
DATED
FILM
8-88
DTIC